PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What is the position of equilibrium of the center?

45. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A fifty-pound cannon-ball is projected vertically upward with a velocity of 300 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A man raises 1 chicken the first year; 6, the second; 35, the third; 180, the fourth; 921, the fifth; 4626, the sixth; 23215, the seventh; 116160, the eighth; and so on. How many does he raise the 20th year, and how many in the twenty years?

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We easily find by inspection $U_{x+1}-5U_x=\dfrac{4^{\frac{x+1}{2}}-1}{3}$, or $\dfrac{4^{\frac{x+2}{2}}-1}{3}$, according as x is odd or even. Integrating and reducing, we have

$$U_x=\tfrac{1}{4}[5^x+4\times5^{x-2}+4^2\times5^{x-4}+\text{etc.}-\frac{4^{\frac{x+1}{2}}-1}{3}\text{ or }\frac{4^{\frac{x+2}{2}}-1}{3}].$$

$$\text{Summing, } S_x=\tfrac{1}{16}[5^{x+1}+4\times5^{x-1}+4^2\times5^{x-3}+\text{etc.}-\frac{23\times4^{\frac{x+2}{2}}-12x-47}{9},$$

$$\text{or }\frac{11\times4^{\frac{x+3}{2}}-12x-47}{9}].$$

Putting $x=20$, and performing operations indicated, we have,
$U_{20}=28,383,163,779,300$, and $S_{20}=35,478,954,491,110$.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.**

The numbers in the problem may be represented under the following form :

1	6	35	180	921	4626
$5\times0+1$,	$5\times1+1$,	$5\times6+5$,	$5\times35+5$,	$5\times180+21$,	$5\times921+21$,

23215	116160	
$5\times4626+85$,	$5\times23215+85$,	etc.

The general term of the numbers 1, 5, 21, 82, etc., is $\frac{1}{3}(4^x-1)$, as can be easily found by Finite Differences. Expressing the $(2x-1)$th term of the above series by $F(2x-1)$, we have, by Finite Differences, $F(2x-1)=C.5^{2x-1}+C_1.4^x+C_2$. Substituting for x successively 1, 2, 3, we have the three equations: $5C+4C_1+C_2=1$, $125C+16C_1+C_2=35$, $3125C+64C_1+C_2=921$, whence $C=25/84$, $C_1=-1/7$, $C_2=1/12$.

$$\therefore\ F(2x-1)=\frac{5^{2x+1}}{84}-\frac{4^x}{7}+\tfrac{1}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{I}).$$

To find $F(2x)$, multiply $F(2x-1)$ by 5 and add $\frac{1}{3}(4^x-1)$, thus,

$$F(2x)=\frac{5^{2x+2}}{84}-\tfrac{8}{21}.4x+\tfrac{1}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{II}).$$

By summing the geometrical series $5^3+5^5+\ldots\ldots\ 5^{2x-1}$, $5^4+5^6+\ldots\ldots\ 5^{2x+2}$, $4+4^2+4^3+\ldots\ldots 4^x$, we find

$$\sum F(2x-1)=\frac{5^{2x+3}}{2016}-\frac{4^{x+1}}{21}+\tfrac{1}{12}x+\tfrac{259}{2016},\text{ and}$$

$$\sum F(2x)=\frac{5^{2x+4}}{2016}-\tfrac{8}{63}.4^{x+1}+\tfrac{1}{12}x+\tfrac{399}{2016}.$$

Consequently $$\sum_{x=1}^{x=2n-1}(x)=\frac{5^{2n+2}}{336}-\tfrac{5}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{35}{144}\ \ldots\ldots\ldots\ldots(\text{III});$$

$$\sum_{x=1}^{x=2n}(x)=\frac{5^{2n+3}}{336}-\tfrac{11}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{47}{144}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{IV}).$$

The formulae I and III are to be employed for an odd number of terms, and II and IV for an even one. Thus, $F(20)=\dfrac{5^{22}}{84}-\tfrac{8}{21}.4^{10}+\tfrac{1}{12}=2838316377930\mathrm{O}$;

$$\sum F(20)=\frac{5^{23}}{336}-\tfrac{11}{63}.4^{11}+\tfrac{10}{6}+\tfrac{47}{144}=35478954491110.$$

III. Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.

1. Write out to "n" terms the series : $1, 5, 25, 125, 625, \ldots\ldots\ldots 5^{n+1}$.
2. Begin at 3rd term and write the series : $4, 20, 100, \ldots\ldots\ldots 4.5^{n-3}$.
3. Begin at 5th term and write the series : $16, 4^2.5, 4^2.5^{n-5}$.

. . . .

4. Begin at $(n-1)$th term and write the series : $4^{\frac{n-1}{2}}, 4^{\frac{n-2}{2}}.5$, n being even.
5. Begin at nth term and write the series : $4^{\frac{n-1}{2}}$, n being odd.

The nth term in the required series is the sum of all the numbers in the nth term of the above arrangement plus all that precede it. Denote the sum by s, then, if n is even,

$$s=\left\{\begin{array}{l} 1+5+25\ldots\ldots\ldots+5^{n-1} \\ +4+4.5\ldots\ldots+4.5^{n-3} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ +\qquad\qquad 4^{\frac{n-2}{2}}+4^{\frac{n-1}{2}}.5 \end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}\ldots\ldots\ldots\frac{4^{\frac{n-2}{2}}(5^2-1)}{4} \quad (1).$$

If n is odd,

$$s=\left\{\begin{array}{l} 1+5+35\ldots\ldots\ldots\ldots 5^{n-1} \\ 4+4.5\ldots\ldots\ldots\ldots 4.5^{n-3} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ +\qquad\qquad\qquad 4^{\frac{n-1}{2}} \end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}\ldots\ldots\ldots 4^{\frac{n-1}{2}}\frac{(5-1)}{4} \quad (2).$$

(1) finally becomes : $\frac{1}{84}\{5^{n+2}+7-8.4^{\frac{n+2}{2}}\}$ . . . . (3),

and (2) becomes $\frac{1}{84}\{5^{n+2}+7-12.4^{\frac{n+1}{2}}\}$ . . . . (4).

(3) gives the even terms; (4) gives the odd terms. To sum the series :—

1st term by $(4)=\frac{1}{84}(5^3+7-12.4)$
2nd term by $(3)=\frac{1}{84}(5^4+7-8.4^2)$
3rd term by $(4)=\frac{1}{84}(5^5+7-12.4^2)$
4th term by $(3)=\frac{1}{84}(5^6+7-8.4^3)$
5th term by $(4)=\frac{1}{84}(5^7+7-12.4^3)$
. . . .

nth term by $(3)=\frac{1}{84}(5^{n+2}+7-8.4^{\frac{n+2}{2}})$, n being even,

nth term by $(4)=\frac{1}{84}(5^{n+2}+7-12.4^{\frac{n+1}{2}})$, n being odd.

Denote the sum by S, then, *n being even*,

$$S = \frac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-11\frac{(4^{\frac{n+4}{2}}-16)}{3}\right\} \ldots\ldots (5).$$

Similarly, n being odd,

$$S = \frac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-\frac{20.4^{\frac{n+3}{2}}-176}{3}\right\} \ldots\ldots (6).$$

By (3), the 20th term is: $\frac{1}{84}\{5^{22}+7-8.4^{11}\}=28,383,163,779,300.$

By (5), the twenty terms are: $\frac{1}{84}\{\frac{5^{23}-125}{4}+140-11\frac{(4^{12}-16)}{3}\}$

$=35,478,954,491,110.$

IV. Solution by the PROPOSER.

Let it be required to sum to n terms and find the nth term of the series:

$1+6x+35x^2+180x^3+921x^4+4626x^5+23215x^6+116160x^7+\ldots\ldots$

Let the scale of relation be denoted by m, n, p, q.

$\therefore 921x^4=180qx^3+35px^2+6nx+m \ldots\ldots (1).$

$4626x^3=921qx^4\times 180px^3+35nx^2+6mx \ldots\ldots (2).$

$23215x^6=4626qx^5+921px^4+180nx^3+35mx^2 \ldots\ldots (3).$

$116160x^7=23215qx^6+4626px^5+921nx^4+180mx^3 \ldots\ldots (4).$

$\therefore m=20x^4,\ n=-24x^3,\ p=-x^2,\ q=6x.$

Since the series has a quadruple scale of relation it must be composed of the sum of four geometrical series. The ratios of these series will be the roots of the biquadratic equation

$r^4=6xr^3-x^2r^2-24x^3r+20x^4 \ldots\ldots (5).$

$\therefore r_1=2x,\ r_2=-2x,\ r_3=5x,\ r_4=x.$

Let a_1, a_2, a_3, a_4 be the first terms of these sets of series; then

$a_1+a_2+a_3+a_4=1 \ldots\ldots (6).$

$a_1r_1+a_2r_2+a_3r_3+a_4r_4=2a_1-2a_2+5a_3+a_4=6 \ldots\ldots (7).$

$a_1r_1{}^2+a_2r_2{}^2+a_3r_3{}^2+a_4r_4{}^2=4a_1+4a_2+25a_3+a_4=35 \ldots\ldots (8).$

$a_1r_1{}^3+a_2r_2{}^3+a_3r_3{}^3+a_4r_4{}^3=8a_1-8a_2+125a_3+a_4=180 \ldots\ldots (9).$

$\therefore a_1=-2/3,\ a_2=2/21,\ a_3=125/84,\ a_4=1/12.$

Hence the series are:

$-2/3-4x/3-8x^2/3-16x^3/3-32x^4/3-64x^5/3-\ldots\ldots (10).$

$2/21-4x/21+8x^2/21-16x^3/21+32x^4/21-64x^5/21+\ldots\ldots (11).$

$125/84+625x/84+3125x^2/84+15625x^3/84$

$+78125x^4/84+390625x^5/84+\ldots\ldots (12).$

$1/12+x/12+x^2/12+x^3/12+x^4/12+x^5/12+\ldots\ldots\ldots\ldots\ldots(13).$

Let $A_n{}^1$, $A_n{}^2$, $A_n{}^3$, $A_n{}^4$, $S_n{}^1$, $S_n{}^2$, $S_n{}^3$, $S_n{}^4$ represent the nth terms, and the sum of n terms of the series (10), (11), (12), (13). Then,

$$A_n{}^1=-\tfrac{2}{3}(2x)^{n-1},\ A_n{}^2=\tfrac{2}{21}(\pm 2x)^{n-1},\ A_n{}^3=\tfrac{125}{84}(5x)^{n-1},\ A_n{}^4=\tfrac{1}{12}x^{n-1},$$

$$S_n{}^1=-\tfrac{2}{3}\left(\frac{2^n x^n-1}{2x-1}\right),\ S_n{}^2=\tfrac{2}{21}\left(\frac{\pm 2^n x^n-1}{-2x-1}\right),\ S_n{}^3=\tfrac{125}{84}\left(\frac{5^n x^n-1}{5x-1}\right),$$

$$S_n{}^4=\tfrac{1}{12}\left(\frac{x^n-1}{x-1}\right).$$

Let A_n, S_n, be the nth term and the sum of n terms of the original series.

$$\therefore\ A_n=\tfrac{1}{84}\{5^{n+2}+7(1-2^{n+2})\mp 2^{n+2}\}x^{n-1}.$$

$$S_n=\tfrac{1}{84}\left\{\frac{125(5^n x^n-1)}{5x-1}-\frac{56(2^n x^n-1)}{2x-1}+\frac{8(\pm 2^n x^n-1}{-2x-1}+\frac{7(x^n-1)}{x-1}\right\}.$$

The upper sign to be used when n is even. Now let $x=1$, $n=20$, and we will get the required results for the problem. $A_{20}=28383163779300$, the number the twentieth year; $S_{20}=35478954491110$, the number in twenty years.

Also solved by *EDWARD R. ROBBINS.*

65. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at the same price per acre, which they sold for \$286.90. A gained as much per cent. on his part as he had acres, B gained 5-8 as much per cent. on his part as A did, and C lost \$9.10 on the cost of his part; the total net gain was 43 9-20 per cent. How much land did each buy, and what did each receive per acre at the sale?

I. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

Let x, y, and z be the number of acres bought by A, B, and C, respectively. $\therefore\ x+y+z=200\ldots\ldots\ldots\ldots\ldots\ldots(1).$

Since the selling price is \$286.90 and the gain per cent. is 43.45, the cost is \$200. Let m=cost per acre; then mx, my, and mz represent the cost of the shares of A, B, and C, respectively. $\therefore\ m(x+y+z)=200$. $\therefore\ m=1$. $\therefore$ the cost of the share of each=number of acres he bought.

x=A's gain per cent., and $5x/8$=B's gain per cent.

$\therefore\ x+x^2/100+y+5xy/800+z-\$9.10=\$286.90.$

$\therefore\ x^2/100+5xy/800=\$96.$ $\therefore\ 8x^2+5xy=76800.$

$$\therefore\ y=\frac{76800-8x^2}{5x}=\frac{15360}{x}-\frac{8x}{5}\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$